

MATH 504 HOMEWORK 1

Due Friday, January 29.

Problem 1. Suppose that a and b are two sets. Use the axioms to show that their symmetric difference, $a \triangle b$, is also a set.

Problem 2. Let y be a set of ordinals.

- (1) If y is nonempty, show that the \in -minimal element in y is unique.
- (2) Show that $\bigcup y$ is an ordinal.

For the following assume that $\alpha, \beta, \gamma, \delta, \xi$ are ordinals.

Problem 3. Show that $\alpha < \beta$ implies that $\gamma + \alpha < \gamma + \beta$ and $\alpha + \gamma \leq \beta + \gamma$. Give an example to show that \leq cannot be replaced with $<$. Also, show that

$$\alpha \leq \beta \rightarrow (\exists! \delta)(\alpha + \delta = \beta).$$

Problem 4. Show that if $\gamma > 0$, then $\alpha < \beta$ implies that $\gamma \cdot \alpha < \gamma \cdot \beta$ and $\alpha \cdot \gamma \leq \beta \cdot \gamma$. Give an example to show that \leq cannot be replaced with $<$. Also, show that

$$(\alpha \leq \beta \wedge \alpha > 0) \rightarrow (\exists! \delta, \xi)(\xi < \alpha \wedge \alpha \cdot \delta + \xi = \beta).$$

Problem 5. Verify that ordinal exponentiation satisfies $\alpha^{\beta+\gamma} = \alpha^\beta \cdot \alpha^\gamma$ and $(\alpha^\beta)^\gamma = \alpha^{\beta \cdot \gamma}$.

Problem 6. Show in ZF^- (i.e. the ZF axioms minus Foundation) that for any set X the following are equivalent:

- (a) X can be well ordered,
- (b) There is a $C : (\mathcal{P}(X) \setminus \{0\}) \rightarrow X$ such that $\forall Y \subset X (Y \neq \emptyset \rightarrow C(Y) \in Y)$.

Hint for (b) \rightarrow (a): Fix $p \neq X$, and let $C(Y) = p$ if $Y \notin \mathcal{P}(X) \setminus \{0\}$. Define by transfinite recursion,

$$F(\alpha) = C(X \setminus \{F(\xi) \mid \xi < \alpha\}).$$